

“SINGULARITIES” IN SPACETIMES WITH DIVERGING HIGHER-ORDER CURVATURE INVARIANTS

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After reviewing the definitions of classical and quantum singularities, it is shown by example that if zeroth-order curvature invariants are regular, a diverging higher-order curvature invariant does not necessarily imply the existence of a classical or a quantum singularity.

1. Introduction

A spacetime (M, g) is a smooth, C^∞ , paracompact, connected Hausdorff manifold M with a Lorentzian metric g . Here we present three spacetimes^{1–3} with regular zeroth-order curvature invariants but diverging higher-order invariants and illustrate with one spacetime³ (the Musgrave-Lake ST) that such a divergence does not necessarily foretell the existence of a “singularity” using the usual definitions.

2. Singularity Definitions

2.1. Classical Singularities

A classical singularity is indicated by incomplete geodesics or incomplete paths of bounded acceleration^{4,5} in a maximal spacetime. Since, by definition, a spacetime is smooth, all irregular points (singularities) have been excised; a singular point is a boundary point of the spacetime. There are 3 types of singularities:⁶ quasi-regular (a mild, topological type), non-scalar curvature (diverging tidal forces on curves ending at the singularity; finite tidal forces on some nearby curves) and scalar curvature (diverging scalars – usually only consider C^0 scalar polynomial (s.p.) invariants).

2.2. Quantum Singularities

A spacetime is QM (quantum-mechanically) nonsingular if the evolution of a test scalar wave packet, representing the quantum particle, is uniquely determined by the initial wave packet, manifold and metric, without having to put boundary conditions at the singularity.⁷ Technically, a static ST is QM-singular if the spatial portion of the relevant wave operator, here the Klein-Gordon operator, is not essentially self-adjoint⁸ on $C_0^\infty(\Sigma)$ in $L^2(\Sigma)$ where Σ is a spatial slice.

3. Spacetimes with Diverging Higher-Order Curvature Invariants

As Musgrave and Lake say, “curvature invariants alone are not sufficient to probe the ‘physics’ of the solution.”³

3.1. *Kinnersley ‘photon rocket’*

Bonnor¹ analyzed the Kinnersley⁹ ‘photon rocket’ which has two-metric functions, the mass $m = m(u)$ and the acceleration $a = a(u)$, both functions of the radial null coordinate u . He found that $a(u)$ does not enter any zeroth-order s.p. curvature scalars, but it does enter into differential invariants. Thus, a singular acceleration ‘singularity’ would not show up on regular curvature invariants but are crucial for an adequate physical picture and predict a true physical singularity since they indicate incomplete, inextendible null geodesics.

3.2. *Siklos Whimper Spacetimes*

Siklos² in 1976 considered the so-called “whimper” STs (“not with a bang, but with a whimper” as the poet T.S. Elliot wrote¹⁰). These STs are geodesically incomplete and inextendible and thus classically singular. They possess C^0 non-scalar curvature singularities; all zeroth-order s.p. invariants are regular. However, Siklos did find these STs to have diverging first-order curvature invariants.

3.3. *Musgrave-Lake Spacetimes*

Musgrave-Lake STs³ are static and spherically-symmetric with metric

$$ds^2 = -(1 + r^{n+3/2})dt^2 + (1 + r^{n+3/2})dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta)d\phi^2 \quad (1)$$

where the coordinates have their usual ranges and the parameter $n = 1, 2, 3, 4$. They have an anisotropic matter distribution and obey all energy conditions (weak, strong, dominate). Physically, they can be interpreted as a “thick shell” with a density and pressure that approach zero as $r \rightarrow 0$ or as $r \rightarrow \infty$. All C^0 s.p. curvature invariants vanish at $r = 0$. Differential invariants up to order $(n - 1)$ are regular at $r = 0$ while n^{th} order differential invariants diverge at $r = 0$. However, there are no incomplete geodesics and hence no classical singularity^a in the usual sense. Observers following timelike and null geodesic paths feel nothing untoward at $r = 0$. In particular, tidal forces do not diverge.

The quantum singularity structure of the Musgrave-Lake spacetime was also studied. The massive Klein-Gordon equation was solved, variables separated, and radial solutions approximated near $r = 0$. Both radial solutions are square integrable, but

^a Actually, for the $n = 1$ case, the tangential pressure is not C^1 and this can be considered the physical reason why the first-order differential invariants diverge.³

one does not form a proper solution of the three dimensional wave equation for the spatial wave function. This can be seen by combining each radial solution with the $l = 0, m = 0$ angular spherical harmonic, integrating over a spherical volume, and finding a nonzero value for one integral. Thus, the Musgrave-Lake spacetime is quantum mechanically non-singular.

4. Conclusions

Spacetimes with higher-order diverging invariants have interesting "singularity" structure. Further study of these and other cases is warranted.

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